

# ON THE CYLINDRICAL GRAD-SHAFRANOV EQUATION

V. S. BESKIN\* and E. E. NOKHRINA

*I.E.Tamm Theoretical Department, P.N.Lebedev Physical Institute, Moscow, Russia*

*\*E-mail: beskin@lpi.ru*

*www.tamm.lpi.ru*

The goal of this presentation is in paying attention to the 1D cylindrical version of the Grad-Shafranov (GS) equation. In our opinion, this approach is more rich than classical self-similar ones, and more suitable for astrophysical jets we observe. In particular, it allows us describing the central (and, hence, the most energetic) part of the flow.

*Keywords:* GS equation, jets, YSO, AGN

## 1. Introduction

An activity of many compact objects – Active Galactic Nuclei (AGNs), Young Stellar Objects (YSOs), microquasars – is associated with the highly collimated jets. These jets are thought to be a natural outlet of an excess angular momentum of a central object and accreting matter [1]. The latest observations indicating the jet rotation in AGNs [2] and YSOs [3] support this idea. The most attractive model for such outflows is the MHD one [1, 4, 5].

Of course, the main question within this model is the collimation itself [4–8]. We assume here that collimation is due to a finite external gas and/or magnetic pressure [9–11]. Indeed, proposing that it is the external magnetic field  $B_{\text{ext}} \sim 10^{-6}$  G that plays the main role in the collimation, we obtain  $r_{\text{jet}} \sim R_{\text{in}} (B_{\text{in}}/B_{\text{ext}})^{1/2}$ . Here  $r$  is the distance from the rotational axis, and the subscripts 'in' correspond to the values in the vicinity of the central object. The similar evaluation can be obtained for external pressure  $p_{\text{ext}} \sim B_{\text{ext}}^2/8\pi$ . E.g., for YSOs ( $B_{\text{in}} \sim 10^3$  G,  $R_{\text{in}} \sim R_{\odot}$ ) we obtain  $r_{\text{jet}} \sim 10^{15}$  cm, in agreement with observational data. Accordingly, for AGNs ( $B_{\text{in}} \sim 10^4$  G,  $R_{\text{in}} \sim 10^{13}$  cm) we have  $r_{\text{jet}} \sim 1$  pc. It means that the external media may indeed play important role in the collimation process.

The internal structure of cylindrical jets was considered both for non-relativistic [12, 13] and relativistic [9, 14–19] flows. In particular, it was shown that for constant angular velocity of plasma  $\Omega_F$  it is impossible to obtain reasonable solution with total zero electric current [9], but it can be constructed if the angular velocity vanishes at the jet boundary and if the external pressure is not equal to zero [11, 17].

Another result was obtained for relativistic and

non-relativistic cylindrical flows [13, 15, 16, 20] is that the poloidal magnetic field  $B_p$  has a jet-like form

$$B_p = \frac{B_0}{1 + r^2/r_{\text{core}}^2}, \quad (1)$$

where  $r_{\text{core}} = v_{\text{in}}\gamma_{\text{in}}/\Omega$ . But this relation corresponds to a very slow (logarithmic) growth of the magnetic flux function:  $\Psi(r) \propto \ln r$ . It means that if the jet core contains only a small part of the total magnetic flux  $\Psi_0$ , the jet boundary is to locate exponentially far from the axis, magnetic field being too weak to be in equilibrium with the external media. In what follows we'll try to resolve this contradiction.

Thus, we consider the following model: the flow crosses all the critical surfaces while the effects of the external media are negligible. It allows us to use standard values of integrals of motion. As the supersonic wind expands, its pressure becomes comparable with the external gas and/or magnetic pressure. The interaction of a flow with external media results in well collimated jet which can be described by 1D cylindrical equations.

## 2. Basic equations

### 2.1. Relativistic flow

For cylindrical flow one can write down electric and magnetic fields as well as the four-velocity of a plasma  $\mathbf{u}$  in standard form

$$\mathbf{B} = \frac{\nabla\Psi \times \mathbf{e}_\varphi}{2\pi r} - \frac{2I}{rc}\mathbf{e}_\varphi, \quad \mathbf{E} = -\frac{\Omega_F}{2\pi c}\nabla\Psi, \quad (2)$$

$$\mathbf{u} = \frac{\eta}{n}\mathbf{B} + \gamma(\Omega_F r/c)\mathbf{e}_\varphi. \quad (3)$$

Here  $n$  is the concentration in the comoving reference frame, and  $\gamma^2 = \mathbf{u}^2 + 1$  is the Lorentz-factor. In other words, it is convenient to express all the values

in terms of magnetic flux  $\Psi$  and total electric current  $I$ , the angular velocity of plasma  $\Omega_F$  and the particle to magnetic flux ratio  $\eta$  being constant on the magnetic surfaces:  $\Omega_F = \Omega_F(\Psi)$ ,  $\eta = \eta(\Psi)$ . Accordingly, the trans-field GS equation can be rewritten as [21]

$$\begin{aligned} \frac{1}{r} \frac{d}{dr} \left( \frac{A}{r} \frac{d\Psi}{dr} \right) + \frac{\Omega_F}{c^2} \left( \frac{d\Psi}{dr} \right)^2 \frac{d\Omega_F}{d\Psi} \\ + \frac{32\pi^4}{r^2 \mathcal{M}^2 c^4} \frac{d}{d\Psi} \left( \frac{G}{A} \right) \\ - \frac{64\pi^4 \mu^2}{\mathcal{M}^2} \eta \frac{d\eta}{d\Psi} - 16\pi^3 n T \frac{ds}{d\Psi} = 0. \end{aligned} \quad (4)$$

Here  $G = r^2(E - \Omega_F L)^2 + \mathcal{M}^2 L^2 c^2 - \mathcal{M}^2 r^2 E^2$ , the Alfvénic factor is  $A = 1 - \Omega_F^2 r^2 / c^2 - \mathcal{M}^2$ ,

$$\mathcal{M}^2 = \frac{4\pi\mu\eta^2}{n} \quad (5)$$

is the Alfvénic Mach number,  $\mu = m_p c^2 + m_p w$  is the relativistic enthalpy, and the derivative  $d/d\Psi$  acts on the integrals of motion only. Finally, the relativistic Bernoulli equation  $u_p^2 = \gamma^2 - u_\varphi^2 - 1$  has a form

$$\frac{\mathcal{M}^4}{64\pi^4 r^2 c^4} \left( \frac{d\Psi}{dr} \right)^2 = \frac{K}{r^2 A^2 c^4} - \mu^2 \eta^2, \quad (6)$$

where

$$K = r^2 (e')^2 (A - \mathcal{M}^2) + \mathcal{M}^4 r^2 E^2 - \mathcal{M}^4 L^2 c^2, \quad (7)$$

and  $e' = E - \Omega_F L$ . Both equations contain relativistic integrals of motion

$$E(\Psi) = \gamma\mu\eta c^2 + \frac{\Omega_F I}{2\pi}, \quad L(\Psi) = r u_\varphi \mu \eta c + \frac{I}{2\pi}, \quad (8)$$

which, as all other invariants, are to be determined from boundary and critical conditions. E.g., for inner part of a flow  $\Psi \ll \Psi_0$  with zero temperature one can choose  $\Omega_F(\Psi) = \Omega_0$ ,  $\eta(\Psi) = \eta_0$ , and

$$E(\Psi) = \mu\eta_0 \gamma_{in} c^2 + \frac{\Omega_0^2}{4\pi^2} \Psi, \quad L(\Psi) = \frac{\Omega_0}{4\pi^2} \Psi. \quad (9)$$

Multiplying now equation (4) on  $2A d\Psi/dr$  and using equation (6), one can find [17]

$$\begin{aligned} \left[ \frac{(e')^2}{\mu^2 \eta^2 c^4} - 1 + \frac{\Omega_F^2 r^2}{c^2} - A \frac{c_s^2}{c^2} \right] \frac{d\mathcal{M}^2}{dr} = \\ \frac{\mathcal{M}^6 L^2}{Ar^3 \mu^2 \eta^2 c^2} + \frac{\Omega_F^2 r \mathcal{M}^2}{c^2} \left[ 2 - \frac{(e')^2}{A \mu^2 \eta^2 c^4} \right] \\ + \mathcal{M}^2 \frac{e'}{\mu^2 \eta^2 c^4} \frac{d\Psi}{dr} \frac{de'}{d\Psi} + \mathcal{M}^2 \frac{r^2}{c^2} \Omega_F \frac{d\Psi}{dr} \frac{d\Omega_F}{d\Psi} \\ - \mathcal{M}^2 \left( 1 - \frac{\Omega_F^2 r^2}{c^2} + 2A \frac{c_s^2}{c^2} \right) \frac{d\Psi}{dr} \frac{1}{\eta} \frac{d\eta}{d\Psi} \\ - \left[ \frac{A}{n} \left( \frac{\partial P}{\partial s} \right)_n + \left( 1 - \frac{\Omega_F^2 r^2}{c^2} \right) T \right] \frac{\mathcal{M}^2}{\mu} \frac{d\Psi}{dr} \frac{ds}{d\Psi}, \end{aligned} \quad (10)$$

where  $c_s \ll c$  is the sound velocity, and the entropy  $s = s(\Psi)$  is the fifth integral of motion. Together with Bernoulli equation (6) it forms the system of two ordinary differential equations for Mach number  $\mathcal{M}^2$  and magnetic flux  $\Psi$  describing cylindrical relativistic jet. Clear boundary conditions are

$$\Psi(0) = 0, \quad (11)$$

$$P(r_{\text{jet}}) = P_{\text{ext}}, \quad (12)$$

where  $P = B^2/8\pi + p$  is the total pressure. Determining the functions  $\mathcal{M}^2(r)$  and  $\Psi(r)$ , one can find the jet radius  $r_{\text{jet}}$  as well as the profile of the current  $I$ , particle energy, and toroidal component of the four-velocity using standard expressions

$$\frac{I}{2\pi} = \frac{L - \Omega_F r^2 E / c^2}{1 - \Omega_F^2 r^2 / c^2 - \mathcal{M}^2}, \quad (13)$$

$$\gamma = \frac{1}{\mu\eta c^2} \frac{(E - \Omega_F L) - \mathcal{M}^2 E}{1 - \Omega_F^2 r^2 / c^2 - \mathcal{M}^2}, \quad (14)$$

$$u_\varphi = \frac{1}{\mu\eta r c} \frac{(E - \Omega_F L) \Omega_F r^2 / c^2 - L \mathcal{M}^2}{1 - \Omega_F^2 r^2 / c^2 - \mathcal{M}^2}. \quad (15)$$

## 2.2. Nonrelativistic flow

In the nonrelativistic limit electric and magnetic fields are determined by general expressions (2). On the other hand, equation (3) can be rewritten as

$$\mathbf{v} = \frac{\eta_n}{\rho_m} \mathbf{B} + \Omega_F r \mathbf{e}_\varphi, \quad (16)$$

where  $\rho_m = m_p n$  is the mass density and  $\eta_n(\Psi)$  is nonrelativistic particle to magnetic flux ratio. Accordingly, nonrelativistic fluxes of energy  $E_n$  and  $z$  component of the angular momentum  $L_n$  are

$$E_n(\Psi) = \frac{\Omega_F I}{2\pi c \eta_n} + \frac{v^2}{2} + w, \quad (17)$$

$$L_n(\Psi) = \frac{I}{2\pi c \eta_n} + v_\varphi r \sin \theta. \quad (18)$$

Further, algebraic relations (13)–(15) can be rewritten as

$$\frac{I}{2\pi} = c \eta_n \frac{L_n - \Omega_F r^2}{1 - \mathcal{M}^2}, \quad (19)$$

$$v_\varphi = \frac{1}{r} \frac{\Omega_F r^2 - L_n \mathcal{M}^2}{1 - \mathcal{M}^2}, \quad (20)$$

where now

$$\mathcal{M}^2 = \frac{4\pi\eta_n^2}{\rho_m}. \quad (21)$$

As a result, nonrelativistic Bernoulli equation

$$\frac{\mathcal{M}^4}{64\pi^4\eta_n^2} \left( \frac{d\Psi}{dr} \right)^2 = 2r^2(E_n - w) - \frac{(\Omega_F r^2 - L_n \mathcal{M}^2)^2}{(1 - \mathcal{M}^2)^2} - 2r^2 \Omega_F \frac{L_n - \Omega_F r^2}{1 - \mathcal{M}^2}, \quad (22)$$

together with nonrelativistic limit of equation (10)

$$\begin{aligned} & \left[ 2e_n - 2w + \Omega_F^2 r^2 - (1 - \mathcal{M}^2)c_s^2 \right] \frac{d\mathcal{M}^2}{dr} = \\ & \frac{\mathcal{M}^6}{1 - \mathcal{M}^2} \frac{L_n^2}{r^3} - \frac{\Omega_F^2 r}{1 - \mathcal{M}^2} \mathcal{M}^2 (2\mathcal{M}^2 - 1) \\ & + \mathcal{M}^2 \frac{d\Psi}{dr} \frac{de_n}{d\Psi} + \mathcal{M}^2 r^2 \Omega_F \frac{d\Psi}{dr} \frac{d\Omega_F}{d\Psi} \\ & + 2 \left[ e_n - w + \frac{\Omega_F^2 r^2}{2} - (1 - \mathcal{M}^2)c_s^2 \right] \frac{\mathcal{M}^2}{\eta_n} \frac{d\Psi}{dr} \frac{d\eta_n}{d\Psi} \\ & - \mathcal{M}^2 \left[ (1 - \mathcal{M}^2) \frac{1}{\rho_m} \left( \frac{\partial P}{\partial s} \right)_{\rho_m} + \frac{T}{m_p} \right] \frac{d\Psi}{dr} \frac{ds}{d\Psi} \end{aligned} \quad (23)$$

where  $e_n = E_n - \Omega_F L_n$ , determine the structure of nonrelativistic cylindrical flow.

### 3. Advantages

Certainly, the approach under consideration is 1D as well. For this reason, it has some properties similar to another self-similar ones. In particular, one can easily check that the singularity on the fast magnetosonic surface is absent. On the other hand, singularity appears on the cusp surface where the factors in front of  $d\mathcal{M}^2/dr$  in (10) and (23) vanish. Nevertheless, in our opinion, this one-dimensional approach has clear advantages in comparison with the standard self-similar ones [4–7].

First of all, it allows us to use any form of the five integrals of motion. Indeed, the self-similarity of a flow demands definite dependence of invariants which may be not correspond to the real boundary conditions. E.g., for relativistic self-similar flow the angular velocity  $\Omega_F$  is to have the form  $\Omega_F \propto r^{-1}$  [22]. It does not correspond neither to the homogeneous stellar rotation, nor to the Keplerian disk rotation. Moreover, this dependence has the singularity at the rotational axis. Thus, the standard self-similar approach cannot describe the central (and, hence, the most energetic) part of the flow.

Further, classical self-similar approach cannot describe the region of electric current closure. Finally, for relativistic magnetically dominated flow it is more convenient to use first-order equation (10) instead of

second order GS equation for which it is necessary to be careful in taking into account small but important terms  $\sim \gamma^{-2}$ . Indeed, the force balance equation (10) does not contain the leading terms  $\rho_e \mathbf{E}$  and  $\mathbf{j} \times \mathbf{B}/c$  as they are analytically removed using Bernoulli equation. As a result, as

$$\frac{|\rho_e \mathbf{E} + \mathbf{j} \times \mathbf{B}/c|}{|\mathbf{j} \times \mathbf{B}/c|} \sim \frac{1}{\gamma^2}, \quad (24)$$

all the terms in equation (10) are of the same order.

In particular, in the limit  $r \gg r_{\text{core}}$ ,  $\mathcal{M}^2 \gg 1$  equation (10) can be rewritten in the simple form [11]

$$\frac{d}{dr} \left( \frac{\mu \eta \Omega_F r^2}{\mathcal{M}^2} \right) - \frac{\mathcal{M}^2 L^2}{\mu \eta \Omega_F r^3 (\Omega_F^2 r^2 / c^2 + \mathcal{M}^2)} = 0. \quad (25)$$

Without the last term  $\propto L^2(\Psi)$  equation (25) results in the conservation of the value  $H$

$$H = \frac{\Omega_F \eta r^2}{\mathcal{M}^2} = \text{const} \quad (26)$$

was found in [26] for conical magnetic field. It is the conservation of  $H$  that results in the jet-like solution (1). Indeed, as  $\eta(\Psi) \approx \text{const}$  and  $\Omega_F(\Psi) \approx \text{const}$  in the center of a jet, we obtain  $\mathcal{M}^2 \propto r^2$ . Using now the definitions  $\mathcal{M}^2 = 4\pi\eta^2\mu/n$  and  $nu_p = \eta B_p$  (and the condition  $u_p \approx \text{const}$  fulfilled in the very center of a flow), we return to (1). But, as we will see, the term containing  $L^2$  (which appears to be missed previously) can be important [17]. It is this term that can change the jet-like structure in relativistic case.

## 4. Internal structure of cylindrical jets

### 4.1. Relativistic flow

#### 4.1.1. General properties

The solution of equations (6) and (10) depends essentially on the Mach number on the rotational axis  $\mathcal{M}_0^2 = \mathcal{M}^2(0)$  [11]. For  $\mathcal{M}_0^2 \gg \mathcal{M}_{\text{cr}}^2$  where  $\mathcal{M}_{\text{cr}}^2 = \gamma_{\text{in}}^2$

$$\mathcal{M}^2 = \mathcal{M}_0^2 \left( 1 + \frac{r^2}{\gamma_{\text{in}}^2 R_L^2} \right), \quad (27)$$

the poloidal magnetic field corresponding to jet-like solution (1). On the other hand, for  $\mathcal{M}_0^2 \ll \mathcal{M}_{\text{cr}}^2$

$$\mathcal{M}^2 = \mathcal{M}_0^2 \left( 1 + \frac{r}{\gamma_{\text{in}} R_L} \right), \quad \Psi = \frac{\gamma_{\text{in}} \Psi_0}{2\mathcal{M}_0^2 \sigma} \left( \frac{r}{R_L} \right)^2. \quad (28)$$

Here  $R_L = c/\Omega_F(0)$ , and  $\sigma = \Omega_0^2 \Psi_0 / 8\pi^2 c^2 \mu \eta_0$  is the Michel magnetization parameter [23] ( $\gamma \approx \sigma$  for particle dominated flow  $W_{\text{part}} \approx W_{\text{em}}$ ). It means that  $B_p \approx \text{const}$ , i.e., the solution has no jet-like form.

As was already stressed, the solution (27) cannot be realized in the presence of the external media. Hence, for any finite pressure  $P_{\text{ext}}$  magnetic field in the center of cylindrical jet  $B_0 = 4\pi\eta_0\mu\gamma_{\text{in}}/\mathcal{M}_0^2$  cannot be much smaller than  $B_{\text{min}} = 4\pi\eta_0\mu\gamma_{\text{in}}/\mathcal{M}_{\text{cr}}^2$ . It gives

$$B_{\text{min}} = \frac{1}{\sigma\gamma_{\text{in}}} B(R_L), \quad (29)$$

where  $B(R_L) = \Psi_0/\pi R_L^2$ .

#### 4.1.2. Central core

Thus, for external magnetic field  $B_{\text{ext}} > B_{\text{min}}$  the internal structure of a relativistic jet is to be described by relations (28). On the other hand, for  $B_{\text{ext}} < B_{\text{min}}$  the core with  $B_p \approx B_{\text{min}}$  is to be formed in the center of a flow (i.e., for  $r < \gamma_{\text{in}} R_L$ ). In particular, for  $\sigma^{-2} B(R_L) < B_{\text{ext}} < B_{\text{min}}$  (and for  $r \gg \gamma_{\text{in}} R_L$ ) the solution can be presented as [18]

$$\mathcal{M}^2 \propto r^\alpha, \quad \Psi \propto r^\beta, \quad (30)$$

the sum being  $\alpha + \beta = 3$ . E.g., for  $B_{\text{ext}} = B_{\text{min}}$  we have  $\alpha = 1, \beta = 2$  (cf. 28), and for  $B_{\text{ext}} = \sigma^{-2} B(R_L)$  we have  $\alpha = 2, \beta = 1$ .

The results presented above were reproduced recently both analytically and numerically. In [18] it was shown that 1D approximation becomes true for paraboloidal outflow at large distances from the equatorial plane  $z \gg \sigma^{2/3} R_L$  where the flow becomes actually cylindrical. Up to the distance  $z = \sigma\gamma_{\text{in}} R_L$  one can use the relations (28), so that the poloidal magnetic field does not depend on  $r$ . The region  $z > \sigma\gamma_{\text{in}} R_L$  corresponds to core-like solution (30). Nevertheless, the transverse dimension of a jet remains parabolic:  $r_{\text{jet}} \propto z^{1/2}$ . Numerically the scalings (30) were confirmed in [24].

Remember that the existence of cylindrical core with  $r_{\text{core}} \sim \gamma_{\text{in}} R_L$  was predicted in many papers [14, 26], but magnetic flux  $\Psi_{\text{core}} = \pi r_{\text{core}}^2 B_{\text{min}}$  inside the core was unknown up to now. As we see, in relativistic case the central core contains only a small part of the magnetic flux:

$$\frac{\Psi_{\text{core}}}{\Psi_0} \approx \frac{\gamma_{\text{in}}}{\sigma}. \quad (31)$$

Nevertheless, as  $\beta > 0$ , such core-like flow can exist in the presence of external media.

#### 4.1.3. Bulk acceleration

As on the fast magnetosonic surface the bulk plasma Lorentz-factor  $\gamma(r_F) \approx \sigma^{1/3}$  (and, hence, here  $W_{\text{part}}/W_{\text{em}} \sim \sigma^{-2/3} \ll 1$ ) [23, 30], the additional particle acceleration is possible for  $r > r_F$ . Using equation (14) and relation  $\alpha + \beta = 3$  one can find that in all region  $B_{\text{ext}} > \sigma^{-2} B(R_L)$  ( $z < \sigma^2 R_L$  for parabolic flow) the Lorentz-factor for  $r > \gamma_{\text{in}} R_L$  can be determined as

$$\gamma \approx r/R_L. \quad (32)$$

Accordingly, one can write down [17]

$$\frac{W_{\text{part}}}{W_{\text{em}}} \sim \frac{1}{\sigma} \left[ \frac{B(R_L)}{B_{\text{ext}}} \right]^{1/2}. \quad (33)$$

It means that for  $B_{\text{ext}} \sim \sigma^{-2} B(R_L)$  ( $z \sim \sigma^2 R_L$  for parabolic flow) where the transverse jet dimension  $r_{\text{jet}} \sim \sigma R_L$  almost the full energy transformation from the Poynting to particle energy flux can be realized. In particular, for the particle moving along parabolic magnetic field line one can obtain

$$\gamma(z) \propto (z/R_L)^{1/2}. \quad (34)$$

This scaling was confirmed numerically as well [27, 28].

It is necessary to stress that relation (32) takes place only if one can neglect the curvature of magnetic surfaces. Indeed, for magnetically dominated case in the limit  $r \gg r_F$  the leading terms in 2D GS equation can be rewritten in the simple form [18]

$$-\frac{1}{2} \mathbf{n} \cdot \nabla(B_p^2) - \frac{B_\varphi^2}{R_c} + \frac{B_\varphi^2 - \mathbf{E}^2}{r} (\mathbf{n} \cdot \mathbf{e}_r) = 0. \quad (35)$$

Here  $R_c$  is the (poloidal) curvature radius of magnetic surfaces, and  $\mathbf{n} = \nabla\Psi/|\nabla\Psi|$ . Neglecting now the curvature term and using standard relations  $B_\varphi \approx B_p r/R_L$  and  $B_\varphi^2 - \mathbf{E}^2 \approx B_\varphi^2/\gamma_{\text{in}}^2$  resulting from (2) and (6), we return to (32). On the other hand, if the curvature is important, then one can neglect the first term in (35), and we obtain

$$\gamma \approx (R_c/r)^{1/2}. \quad (36)$$

This scaling taking place for split-monopole geometry outside the fast magnetosonic surface corresponds to  $\gamma \approx \sigma^{1/3} \ln^{1/3}(r/r_F)$  [29, 30]. Remember that for  $r < r_F$  we have "linear" acceleration (32). Thus, the effective particle acceleration can take place only if  $r_{\text{jet}} \sim \sigma R_L$ , and if the curvature of magnetic surfaces is not important.

#### 4.1.4. In the center of the self-similar domain

The approach under consideration allows us matching the self-similar solution to the rotational axis. Indeed, for relativistic self-similar invariants

$$\Omega_F(\Psi) = \Omega_0(\Psi/\Psi_b)^{-b}, \quad (37)$$

$$E(\Psi) = E_0(\Psi/\Psi_b)^{1-2b}, \quad (38)$$

$$L(\Psi) = L_0(\Psi/\Psi_b)^{1-b}, \quad (39)$$

$$\eta(\Psi) = \eta_0(\Psi/\Psi_b)^{1-2b}, \quad (40)$$

the solution of the 2D GS equation for  $\Psi > \Psi_b$  has the form  $\Psi(\rho, \theta) = \rho^{1/b}\Theta(\theta)$ , where  $\rho$  is the spherical radius. Hence, far from the equatorial plane where  $z \gg r$  ( $\theta \ll 1$ ,  $\rho \approx z$ ) one can write down

$$\Psi(\rho, \theta) = \mathcal{A}\rho^{1/b}\theta^a. \quad (41)$$

As a result, the cylindrical radius of the boundary  $\Psi = \Psi_b$  can be written as

$$r_b(z) = \mathcal{A}^{-1/a}\Psi_b^{1/a}z^{1-1/ab}. \quad (42)$$

Let us consider now the central part of a flow  $\Psi < \Psi_b$ . If again  $\theta \ll 1$ , one can integrate 1D cylindrical equations (6) and (10) considering  $z \approx \rho$  as a parameter. Assuming that  $\Omega_F = \Omega_0$  and  $\eta = \eta_0$  for  $\Psi < \Psi_b$  and using solution (28) we have for  $\mathcal{M}_b^2(z) = \mathcal{M}^2(r_b)$

$$\mathcal{M}_b^2(z) = \frac{8\pi^2\eta_0\mu}{aR_L\mathcal{A}^{3/a}\Psi_b^{1-3/a}}z^{3-3/ab}. \quad (43)$$

On the other hand, for  $\Psi > \Psi_b$  ( $r > r_b$ ) one can seek the solution in a form  $\mathcal{M}^2(r) = \mathcal{M}_b^2(r/r_b)^\varepsilon$ . As a result, equations (6), (10) give

$$a = 2, \quad \varepsilon = 3 - 6b. \quad (44)$$

Substituting now  $r \approx z\theta$ , we obtain

$$\mathcal{M}^2 = \mathcal{C}\theta^\varepsilon. \quad (45)$$

The coefficient  $\mathcal{C} \propto \mathcal{M}_b^2(z)z^\varepsilon/r_b^\varepsilon(z)$ , in agreement with self-similar property, does not depend on  $z$ .

## 4.2. Nonrelativistic flow

### 4.2.1. Central core

For nonrelativistic case in the central part of a flow one can use general expressions (9)

$$E_n(\Psi) = \frac{v_{in}^2}{2} + i_0 \frac{\Omega_0^2}{4\pi^2 c \eta_0} \Psi, \quad L_n(\Psi) = i_0 \frac{\Omega_0}{4\pi^2 c \eta_0} \Psi, \quad (46)$$

non-dimensional current  $i_0 = j/j_{GJ}$  depending now on the angular velocity  $\Omega_F$ . For  $\Omega_F \ll \Omega_{cr}$ , where

$$\Omega_{cr} = \frac{v_{in}}{R_{in}} \left( \frac{\rho_{in} v_{in}^2}{B_{in}^2 / 8\pi} \right)^{1/2}, \quad (47)$$

corresponding to particle dominated outflow near the star the 2D problem can be solved analytically [20, 25], and we obtain  $i_0 = c/v_{in}$ . For magnetically dominated flow near the origin one can write down [10]

$$i_0 \approx c/v_{in} (\Omega_F/\Omega_{cr})^{-2/3}. \quad (48)$$

Nevertheless, for  $\Psi < \Psi_{in}$ , where

$$\Psi_{in} = \frac{4\pi^2 v_{in}^3 \eta_0}{i_0 \Omega_0^2}, \quad (49)$$

the flow remains particle dominated:  $E_n \approx v_{in}^2/2$ . Remember that in the nonrelativistic case the flow can pass smoothly the critical surfaces only if  $W_{part}(r_F) \sim W_{em}(r_F)$ . Thus, the flow at large distances is to be particle dominated.

Solving now equations (22) and (23) for sub Alfvénic flow  $\mathcal{M}_0^2 < 1$  we obtain that poloidal magnetic field remains constant inside the jet up to the very boundary. Thus, one can put  $B(0) = B_{ext}$ . But such a flow can exist only in the presence of large enough external magnetic field  $B_{ext} > B(r_F)$ . For ordinary YSOs  $B(r_F) \sim 10^{-1}$  G, so sub Alfvénic flow in a jet cannot be realized.

On the other hand, for super-Alfvénic cold outflow one can find that the term  $\propto L_n^2$  in (23) plays no role. It means that here  $H \approx \text{const}$ , and we return to jet-like solution (1) [16, 20]. But, as was already stressed, in the presence of finite external pressure it is possible if the central core  $r < r_{core} = v_{in}/\Omega$  contains almost all magnetic flux  $\Psi_0$ . This can be realized only for slow rotation  $\Omega_F \ll \Omega_{cr}$ . In this case magnetic field on the axis cannot be much smaller than  $B_{min} = \Psi_0/\pi r_{core}^2$ :

$$B_0 = \frac{B_{min}}{\ln(1 + B_{min}/B_{ext})}. \quad (50)$$

Accordingly,  $\Psi_{core} = \Psi_0/\ln(1 + B_{min}/B_{ext})$ . This structure was reproduced numerically as well [10].

But for fast rotation  $\Omega_F \gg \Omega_{cr}$  the core magnetic flux  $\Psi_{core}$  is much smaller even than the flux  $\Psi_{in}$  (49) within the central part of a flow:

$$\frac{\Psi_{core}}{\Psi_{in}} \approx \frac{i_0 v_{in}}{2c M_0^2} \ll 1. \quad (51)$$

It means that the cold cylindrical flow resulting from the interaction of fast rotating supersonic wind with the external media cannot be realized.

#### 4.2.2. Heating in oblique shock

To resolve this contradiction, one can propose that in real nonrelativistic jets an important role may play the finite temperature. E.g., the additional heating can be connected with the oblique shock near the base of a jet [31, 32]. It is well known that such a shock is needed to explain the emission lines observed in YSOs [33]. This situation is alike the hydrodynamical supersonic flow meeting the wall. This analogy is all the more reasonable as the non-relativistic supersonic outflow is to be particle dominated.

To evaluate the thermal terms in equations (22) and (23) we consider pure hydrodynamical shock wave swifiting spherically symmetric supersonic flow into cylindrical jet. Knowing the swifiting angle, one can determine the entropy jump  $\Delta s$  as a function of particle flux, all other four invariants being the same as in front of a shock. As a result, we found that fastly rotating jet with  $\Omega_F \gg \Omega_{cr}$  heated in a shock is to have core-jet structure (30) with  $\alpha < 2$ ,  $\beta > 0$ . Hence, it can be realized in the presence of external media. Obtained jet parameters ( $T \sim 10^4$  K,  $v_\varphi \sim 10$  km/s at  $r \sim 10$  A.U.) [34] are in agreement with observational data.

#### 4.2.3. In the center of the self-similar domain

The procedure similar to Sect. 4.1.4. for nonrelativistic particle dominated flow gives for self-similar region ( $\Psi > \Psi_b$ ,  $\theta \ll 1$ ,  $E_n \propto \Psi^{-b'}$ ) that  $\mathcal{M}^2 = \mathcal{C}\theta^\varepsilon$ ,  $\varepsilon = 2 - 4b'$ , as in numerical simulation [35].

## 5. Conclusion

Thus, cylindrical GS equation has definite advantages in comparison with standard self-similar ones. Using this approach it was demonstrated that in relativistic case effective particle acceleration can take place only if  $r_{jet} \sim \sigma R_L$ , the curvature of magnetic surfaces playing no role. For nonrelativistic flow we found that the heating in oblique shock near the base of a jet must play the leading role for magnetically dominated flow. In both cases the magnetic flux within the central core was determined.

## 6. Acknowledgments

This work was supported by Russian Foundation for Basic Research (Grant no. 08-02-00749) and Dynasty fund.

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